Turbulent Flows
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Solution to Exercise 10.5

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a:) According to the $k - \varepsilon$ model, the specification of turbulent viscosity is

$$\nu_T = C_\mu k^2/\varepsilon. \tag{1}$$

So in a simple turbulent shear flow, the shear stress is given by

$$|\langle uv\rangle| = \nu_T \frac{\partial \langle U\rangle}{\partial y} = \nu_T \mathcal{S}. \tag{2}$$

Substituting Eq. 1 into Eq. 2, we get

$$\frac{|\langle uv\rangle|}{k} = C_\mu \frac{\mathcal{S}k}{\varepsilon}. \tag{3}$$

For simple turbulent shear flow, the production is

$$\mathcal{P} = |\langle uv\rangle| \frac{\partial U}{\partial y} = |\langle uv\rangle| \mathcal{S}. \tag{4}$$

Substituting Eq. 3 into Eq. 4, we get

$$\mathcal{P} = C_\mu \frac{\mathcal{S}k^2}{\varepsilon}\mathcal{S} = C_\mu \frac{\mathcal{S}^2 k^2}{\varepsilon}, \tag{5}$$

i.e.,

$$\frac{\mathcal{P}}{\varepsilon} = C_\mu \left(\frac{\mathcal{S}k}{\varepsilon}\right)^2 \tag{6}$$

From Eq. 6, we get

$$\frac{\mathcal{S}k}{\varepsilon} = \left(\frac{\mathcal{P}}{C_\mu \varepsilon}\right)^{1/2}. \tag{7}$$

Substituting Eq. 7 into Eq. 3, we get

$$\frac{|\langle uv\rangle|}{k} = C_\mu^{1/2} \left(\frac{\mathcal{P}}{\varepsilon}\right)^{1/2}, \tag{8}$$

and hence Eq.(10.48) is verified.

b:) The Cauchy-Schwartz inequality (Eq.(3.100)) is

$$-1 \leq \rho_{12} \leq 1, \tag{9}$$

i.e.,

$$\left| \frac{\langle uv \rangle}{\langle u^2 \rangle^{1/2} \langle v^2 \rangle^{1/2}} \right| \leq 1. \tag{10}$$

Substituting Eq. 3 into Eq. 10, we get

$$C_\mu \frac{k^2 \mathcal{S}}{\varepsilon} \leq \langle u^2 \rangle^{1/2} \langle v^2 \rangle^{1/2}, \tag{11}$$

i.e.,

$$C_\mu \leq \frac{\langle u^2 \rangle^{1/2} \langle v^2 \rangle^{1/2}}{k} \frac{1}{\mathcal{S}k/\varepsilon}. \tag{12}$$

For simple shear flow, the $k - \varepsilon$ model yields for the normal stresses $\langle u^2 \rangle = \langle v_2 \rangle = \langle w^2 \rangle = \frac{2}{3}k$. Thus we obtain

$$C_\mu \leq = \frac{2/3}{\mathcal{S}k/\varepsilon}. \tag{13}$$

c:) For a general flow, according to the turbulent viscosity hypothesis, turbulent production $\mathcal{P}$ is equal to

$$\mathcal{P} = \nu_T \mathcal{S}^2, \tag{14}$$

where $\mathcal{S}$ is the characteristic mean rate of strain (see Eq. 5.143 and see page 131 for the definition of $\mathcal{S}$). And furthermore, according to the specification of $\nu_T = C_\mu k^2/\varepsilon$ in the $k - \varepsilon$ model, $\mathcal{P}$ is given by

$$\mathcal{P} = C_\mu \varepsilon \left(\frac{\mathcal{S}k}{\varepsilon} \right)^2, \tag{15}$$

i.e.

$$\frac{\mathcal{P}}{\varepsilon} = C_\mu \left(\frac{\mathcal{S}k}{\varepsilon} \right)^2. \tag{16}$$

So Eq.(10.50) also holds for a general flow.