

Turbulent Flows
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Solution to Exercise 10.5

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a:) According to the $k - \varepsilon$ model, the specification of turbulent viscosity is

$$\nu_T = C_\mu k^2 / \varepsilon. \quad (1)$$

So in a simple turbulent shear flow, the shear stress is given by

$$|\langle uv \rangle| = \nu_T \frac{\partial \langle U \rangle}{\partial y} = \nu_T \mathcal{S}. \quad (2)$$

Substituting Eq. 1 into Eq. 2, we get

$$\frac{|\langle uv \rangle|}{k} = C_\mu \frac{\mathcal{S} k}{\varepsilon}. \quad (3)$$

For simple turbulent shear flow, the production is

$$\mathcal{P} = |\langle uv \rangle| \frac{\partial U}{\partial y} = |\langle uv \rangle| \mathcal{S}. \quad (4)$$

Substituting Eq. 3 into Eq. 4, we get

$$\mathcal{P} = C_\mu \frac{\mathcal{S} k^2}{\varepsilon} \mathcal{S} = C_\mu \frac{\mathcal{S}^2 k^2}{\varepsilon}, \quad (5)$$

i.e.,

$$\frac{\mathcal{P}}{\varepsilon} = C_\mu \left(\frac{\mathcal{S} k}{\varepsilon} \right)^2 \quad (6)$$

From Eq. 6, we get

$$\frac{\mathcal{S} k}{\varepsilon} = \left(\frac{\mathcal{P}}{C_\mu \varepsilon} \right)^{1/2}. \quad (7)$$

Substituting Eq. 7 into Eq. 3, we get

$$\frac{|\langle uv \rangle|}{k} = C_\mu^{1/2} \left(\frac{\mathcal{P}}{\varepsilon} \right)^{1/2}, \quad (8)$$

and hence Eq.(10.48) is verified.

b:) The Cauchy-Schwartz inequality (Eq.(3.100)) is

$$-1 \leq \rho_{12} \leq 1, \quad (9)$$

i.e.,

$$\left| \frac{\langle uv \rangle}{\langle u^2 \rangle^{1/2} \langle v^2 \rangle^{1/2}} \right| \leq 1. \quad (10)$$

Substituting Eq. 3 into Eq. 10, we get

$$C_\mu \frac{k^2 \mathcal{S}}{\varepsilon} \leq \langle u^2 \rangle^{1/2} \langle v^2 \rangle^{1/2}, \quad (11)$$

i.e.,

$$C_\mu \leq \frac{\langle u^2 \rangle^{1/2} \langle v^2 \rangle^{1/2}}{k} \frac{1}{\mathcal{S}k/\varepsilon}. \quad (12)$$

For simple shear flow, the $k - \varepsilon$ model yields for the normal stresses $\langle u^2 \rangle = \langle v^2 \rangle = \langle w^2 \rangle = \frac{2}{3}k$. Thus we obtain

$$C_\mu \leq \frac{2/3}{\mathcal{S}k/\varepsilon}. \quad (13)$$

c:) For a general flow, according to the turbulent viscosity hypothesis, turbulent production \mathcal{P} is equal to

$$\mathcal{P} = \nu_T \mathcal{S}^2, \quad (14)$$

where \mathcal{S} is the characteristic mean rate of strain (see Eq. 5.143 and see page 131 for the definition of \mathcal{S}). And furthermore, according to the specification of $\nu_T = C_\mu k^2/\varepsilon$ in the $k - \varepsilon$ model, \mathcal{P} is given by

$$\mathcal{P} = C_\mu \varepsilon \left(\frac{\mathcal{S}k}{\varepsilon} \right)^2, \quad (15)$$

i.e.

$$\frac{\mathcal{P}}{\varepsilon} = C_\mu \left(\frac{\mathcal{S}k}{\varepsilon} \right)^2. \quad (16)$$

So Eq.(10.50) also holds for a general flow.

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